**Indian Statistical Institute, Bangalore Centre**
B.Math (Hons.) II Year, Second Semester
Mid-Sem Examination - 2013-2014
Graph Theory

Duration: 3 Hours     February 28, 2014     Instructor: C.R.E. Raja

**Answer any five, each question carries 8 marks, total marks: 40**

1. Let $G$ be a graph and consider the following:

   (i) $G$ has an Euler circuit;

   (ii) the edge set of $G$ can be partitioned into cycles;

   (iii) every vertex of $G$ has even degree.

   Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Further, if $G$ is connected, prove that (iii) $\Rightarrow$ (i).

2. (a) If $\delta(G) \geq |G|/2 \geq 3/2$, prove that $G$ has a $H$-cycle (*Marks 5*).

   (b) If $\delta(G) \geq 2$, prove that $G$ has a cycle of length at least $\delta(G) + 1$.

3. (a) If a graph $G$ is not connected, prove that its complement graph is connected.

   (b) A graph of order $n$ is a tree if and only if $\sum d(x) = 2(n-1)$ and $d(x) \geq 1$ for all $x$ (*Marks 5*).

4. Let $G$ be a directed graph, $s \neq t$ be two vertices of $G$ with capacity function $c$. Prove that there is a flow from $s$ to $t$ with maximal flow value and the maximal value is equal to the minimum of the capacities of cuts separating $s$ from $t$.

5. (a) If $f$ is a face of a plane graph $G$ and $H$ is a subgraph of $G$ whose edges are the boundary of $f$ and vertices are the end vertices of these edges, prove that $f$ is also a face of $H$.

   (b) If $G$ is a plane graph with $\delta(G) \geq 5$, prove that $G$ has at least 12 vertices of degree 5 (*Marks 4*).

6. (a) Let $L(G)$ be the line graph of a graph $G$. If $G$ is connected, then prove that $L(G)$ is connected. Is the converse true! Justify your answer (*Marks 4*).

   (b) Let $G = (V, E)$ be a graph and $S, T \subset V$. Then the maximal number of vertex-disjoint $S - T$ paths in $G$ is equal to

$$\min\{|W| \mid W \subset V, \text{ there is no S} - \text{T path in } G - W\}.$$

7. (a) A graph with at least 3 vertices is 2-connected if and only if every pair of vertices lies in a cycle (*Marks 5*).

   (b) If $G_1$ and $G_2$ are two connected subgraphs of $G$ having at least one vertex in common, then $G_1 \cup G_2$ is connected.